This week

1. Section 1.1: functions and their graphs
2. Section 1.2: combining and transforming functions
3. Section 1.3: trigonometry
About this course

- Colstructie = **College** + **Instructie** = “Lectorial”.
- Three midterm tests and one resit. See MyTimeTable for date and time.
- Tests and exercises with MyLabsPlus.
- Examples with *Mathematica*.
- Course schedule, slides and other materials can be found on Blackboard page **2017-20170041-1B: Smart Environments (2017-1B)**, item **Course Materials**, folder **ItE:Introduction to Mathematics and Modeling I**.

Topics of this course

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**Definition**

A function \( f : D \to Y \) is a rule that assigns a unique element \( f(x) \) in \( Y \) to each element \( x \) in \( D \).

- The set \( D \) is the **domain** of \( f \).
- The set \( Y \) is the **codomain** of \( f \).
- The **range** or **image** of \( f \) is the set of all function values \( f(x) \).
- If \( f \) assigns \( y \) to \( x \), then we denote this as \( y = f(x) \) or \( x \mapsto f(x) \).
- The object \( f(x) \) is called the **image of** \( x \) (under \( f \)).
- Synonyms for ‘function’ are **map** or **transformation**.
- Sometimes we use a **diagram**:

![Diagram](x \mapsto f(x))

**The graph of a function**

**Definition**

Let \( D \) and \( Y \) are subsets of \( \mathbb{R} \). The **graph** of a function \( f : D \to Y \) is defined as

\[
\text{graph}(f) = \{(x, f(x)) \mid x \in D\}.
\]

![Graph of a function](x, f(x), f(x))

**Vertical Line Test**

A vertical line intersects the graph of a function in at most one point.
Plotting with Mathematica

**Mathematica**

- **Defining a function:**
  
  \[
  f[x_] := 1/\sqrt{x^2 + 1}
  \]

- **Plotting a function \( f \) with domain \([a, b]\):**
  
  \[
  \text{Plot}[f[x], \{x, a, b\}]
  \]

Empty notebook (Worksheet 1.nb)

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Equality of functions

**Definition**

Two functions \( f \) and \( g \) are **equal** if

1. the domain of \( f \) is equal to the domain of \( g \), and
2. if \( f(x) = g(x) \) for all \( x \) in the domain of \( f \).

- Sometimes the equality of the codomains is also required.
- The method that the function uses to calculate the function value is irrelevant.
Implicitly defined domains and codomains

**Definition**

Let the function $f$ be defined by a formula.

- If the domain of $f$ is not defined explicitly, then the domain consists of all numbers $x$ for which $f(x)$ exists.
- If the codomain of $f$ is not defined explicitly, then the codomain is chosen as large as possible.

**Example:**

Let $f(x) = \sqrt{x - 3}$.

- The expression $\sqrt{x - 3}$ is defined for all $x$ for which $x - 3 \geq 0$, hence $\text{Dom}(f) = [3, \infty)$.
- The codomain is $\mathbb{R}$.

---

**Piecewise defined functions**

If a domain consists of several parts $D_1, D_2, \ldots, D_n$, a function may be defined with different formulas per part:

$$f(x) = \begin{cases} 
F_1(x) & \text{if } x \in D_1, \\
F_2(x) & \text{if } x \in D_2, \\
\vdots & \vdots \\
F_n(x) & \text{if } x \in D_n.
\end{cases}$$

**Example**

The absolute value is defined as

$$|x| = \begin{cases} 
x & \text{if } x \geq 0, \\
-x & \text{if } x < 0.
\end{cases}$$
Monotony

Definition

Let \( f : I \to \mathbb{R} \) be a function defined on an interval \( I \).

1. The function \( f \) is increasing if for all \( x_1, x_2 \in I \) with \( x_1 < x_2 \) we have \( f(x_1) < f(x_2) \).
2. The function \( f \) is decreasing if for all \( x_1, x_2 \in I \) with \( x_1 < x_2 \) we have \( f(x_1) > f(x_2) \).

A function that is decreasing or increasing is called **monotonous**.

Symmetry

Definition

A subset \( D \subset \mathbb{R} \) is symmetric if for all \( x \in D \) also \( -x \in D \).

Example

- \( \mathbb{R} \), \([-1, 1]\) and \( \mathbb{R}\{0\} \) are symmetric.
- \([0, 1]\) en \([-1, 1)\) are not symmetric.

Definition

Let \( D \) be a symmetric subset of \( \mathbb{R} \).

- A function \( f : D \to \mathbb{R} \) is even if \( f(-x) = f(x) \) for all \( x \in D \).
- A function \( f : D \to \mathbb{R} \) is odd if \( f(-x) = -f(x) \) for all \( x \in D \).
Symmetry

- The graph of an even function is symmetric about the vertical axis.
  \[ f(-x) = f(x) \]

- The graph of an odd function is symmetric about the origin.
  \[ f(-x) = -f(x) \]

Exercises

Assignment: **IMM1 - Tutorial 1.1.**
Algebraic combinations

2.1

Addition: \( h(x) = f(x) + g(x) \)  \( f + g \)

Subtraction: \( h(x) = f(x) - g(x) \)  \( f - g \)

Multiplication: \( h(x) = f(x)g(x) \)  \( fg \)

Division: \( h(x) = \frac{f(x)}{g(x)} \)  \( \frac{f}{g} \)

Composition: \( h(x) = f(g(x)) \)  \( f \circ g \)

Composition

2.2

Let \( f: D \to C \) and \( g: E \to D \) be two functions, where the domain of \( f \) is the codomain of \( g \).

The composition of \( f \) and \( g \) is defined as the function \( f \circ g: E \to C \) that assigns the element \( f(g(x)) \) to every \( x \in E \).

Pronounce \( f \circ g \) as “\( f \) after \( g \)”.

Arrow diagram:
Example

Define \( f(x) = \sqrt{x} \) and \( g(x) = x + 1 \). Find \( f \circ g \) and \( g \circ f \).

\[
(f \circ g)(x) = f(g(x)) = \sqrt{x + 1}
\]

\[
(g \circ f)(x) = g(f(x)) = \sqrt{x + 1}
\]

Note that \( f \circ g \neq g \circ f \).

Associativity

Let \( h : F \to E \), \( g : E \to D \) and \( f : D \to C \), then

\[
(f \circ g) \circ h = f \circ (g \circ h).
\]

Usually we omit the parenthesis: \( f \circ g \circ h \).
**Vertical shifting**

**Shifting in y-direction**

The graph of \( f(x) + c \) is obtained from the graph of \( f \) by shifting it upward by \( c \) units if \( c > 0 \), or downward by \(|c|\) units if \( c < 0 \).

**Example**

\[
\begin{align*}
y &= x^2 \\
y &= x^2 + 2 \\
y &= x^2 - 1
\end{align*}
\]

**Vertical scaling**

**Scaling in y-direction**

- The graph of \( cf(x) \) is obtained from the graph of \( f \) by stretching it with a factor of \( c \) units if \( c > 1 \), or shrinking it by \( c \) units if \( 0 < c < 1 \).
- If \( c < 0 \), then the graph is also reflected across the \( x \)-axis.

**Example**

\[
\begin{align*}
y &= \sqrt{x} \\
y &= 2\sqrt{x} \\
y &= \frac{1}{2}\sqrt{x}
\end{align*}
\]
Horizontal shifting

2.7

Shifting in x-direction

The graph of $f(x + c)$ is obtained from the graph of $f$ by shifting it $c$ units to the left if $c > 0$, or to the right by $|c|$ units if $c < 0$.

Example

\[
\begin{align*}
  y &= x^2 \\
  y &= (x + 2)^2 \\
  y &= (x - 1)^2
\end{align*}
\]


Horizontal scaling

2.8

Scaling in x-direction

- The graph of $f(cx)$ is obtained from the graph of $f$ by shrinking it with a factor of $c$ units if $c > 1$, or stretching it with a factor $c$ units if $0 < c < 1$.
- If $c < 0$, then the graph is also reflected across the $y$-axis.

Example

\[
\begin{align*}
  y &= (x - 1)^2 \\
  y &= (2x - 1)^2 \\
  y &= \left(\frac{1}{2}x - 1\right)^2
\end{align*}
\]
Reflections

Reflecting in $x$ and $y$-direction

**Mirroring:** The graph of $f(-x)$ is obtained from the graph of $f$ by reflecting it across the $y$-axis.

**Flipping:** The graph of $-f(x)$ is obtained from the graph of $f$ by reflecting it across the $x$-axis.

Example

- $y = \sqrt{x}$
- $y = \sqrt{-x}$
- $y = -\sqrt{x}$

Exercises

Assignment: IMM1 - Tutorial 1.2.
The number $\pi$

$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

$\pi \approx 3.141592653589793238462643383279502884197169399375105820\ldots$

Radians

**Theorem**

In a sector, the length of the arc is proportional to the angle of the sector and the radius of the circle.

- $L \propto r \theta \quad \Rightarrow \quad L = k r \theta$.
- The constant $k$ depends on the units for measuring angles.
- The **radian** is a unit for angles such that $k = 1$.
- A full circle is $2\pi$ radians.
- If angles are measured in radians then $L = r \theta$. 
Sine, cosine and tangent for acute angles

Triangle $ABC$ is rectangular ($\angle ABC = \pi/2$), angle at $A$ is acute ($0 < \theta < \pi/2$).

\[
\begin{align*}
\cos \theta &= \frac{AB}{AC}, \\
\sec \theta &= \frac{1}{\cos \theta}, \\
\sin \theta &= \frac{BC}{AC}, \\
\csc \theta &= \frac{1}{\sin \theta}, \\
\tan \theta &= \frac{BC}{AB} = \frac{BC}{AC} \cdot \frac{AC}{AB} = \frac{\sin \theta}{\cos \theta}, \\
\cot \theta &= \frac{1}{\tan \theta}.
\end{align*}
\]
Sine and cosine for arbitrary angles

For arbitrary angles, the sine, cosine are defined with the **unit circle**: the circle with center \((0, 0)\) and radius 1.

![Unit Circle Diagram](image)

**Graphs of sine and cosine**

![Sine Graph](image)

![Cosine Graph](image)
Graph of the tangent

\[
\begin{align*}
\tan \left( -\frac{\pi}{4} \right) &= \tan \left( \frac{3\pi}{4} \right) = -1 \\
\tan \left( -\frac{\pi}{2} \right) &= \text{undefined} \\
\tan \left( 0 \right) &= 0 \\
\tan \left( \frac{\pi}{2} \right) &= \text{undefined} \\
\tan \left( \frac{\pi}{4} \right) &= 1 \\
\tan \left( \frac{3\pi}{4} \right) &= -1 \\
\tan \left( \frac{\pi}{3} \right) &= \sqrt{3} \\
\tan \left( \frac{\pi}{6} \right) &= \frac{1}{\sqrt{3}} \\
\end{align*}
\]

Sine, cosine and tangent of special angles

\[
\begin{align*}
\cos \left( \frac{\pi}{4} \right) &= \frac{1}{\sqrt{2}} \\
\sin \left( \frac{\pi}{4} \right) &= \frac{1}{\sqrt{2}} \\
\tan \left( \frac{\pi}{4} \right) &= 1 \\
\cos \left( \frac{\pi}{6} \right) &= \frac{1}{2} \\
\sin \left( \frac{\pi}{6} \right) &= \frac{1}{2} \\
\tan \left( \frac{\pi}{6} \right) &= \frac{1}{\sqrt{3}} \\
\cos \left( \frac{\pi}{3} \right) &= \frac{1}{2} \\
\sin \left( \frac{\pi}{3} \right) &= \frac{1}{2} \\
\tan \left( \frac{\pi}{3} \right) &= \sqrt{3}
\end{align*}
\]
Sine and cosine are **periodic**:

\[
\cos x = \cos(x + 2\pi) \quad \text{for all } x \in \mathbb{R},
\]

\[
\sin x = \sin(x + 2\pi) \quad \text{for all } x \in \mathbb{R}.
\]

Sine and cosine are **congruent**:

\[
\cos x = \sin\left(x + \frac{1}{2}\pi\right) \quad \text{for all } x \in \mathbb{R},
\]

\[
\sin x = \cos\left(x - \frac{1}{2}\pi\right) \quad \text{for all } x \in \mathbb{R}.
\]

Sine and cosine are **symmetric**:

\[
\cos(-x) = \cos x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \cos x \text{ is even},
\]

\[
\sin(-x) = -\sin x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \sin x \text{ is odd},
\]

Sine and cosine are **half-periodic**:

\[
\cos(x - \pi) = \cos(x + \pi) = -\cos x \quad \text{for all } x \in \mathbb{R},
\]

\[
\sin(x - \pi) = \sin(x + \pi) = -\sin x \quad \text{for all } x \in \mathbb{R}.
\]
Pythagoras' theorem

\[ \cos^2 \theta + \sin^2 \theta = \left( \frac{c}{b} \right)^2 + \left( \frac{a}{b} \right)^2 = \frac{c^2}{b^2} + \frac{a^2}{b^2} = \frac{c^2 + a^2}{b^2} = \frac{b^2}{b^2} = 1. \]

Pythagoras' theorem
The law of sines

### Theorem

For any arbitrary triangle with angles $\alpha$, $\beta$, $\gamma$ and edge lengths $a$, $b$, $c$ as defined above, the following equations hold:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

**Proof:**

$$\frac{\sin \alpha}{a} = \frac{h/b}{a} = \frac{h}{ab} = \frac{h/a}{b} = \frac{\sin \beta}{b}.$$

### Sum rule for sine

#### Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ we have

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

**From the law of sines follows**

$$\frac{\sin(\alpha + \beta)}{c} = \frac{\sin \varphi}{a},$$

hence

$$\sin(\alpha + \beta) = \frac{c \sin \varphi}{a} = \frac{c h/b}{a} = \left(\frac{c_1 + c_2}{a}\right) \frac{h}{b}$$

$$= \frac{c_1 h}{b a} + \frac{h c_2}{b a}$$

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$
Sum rule for cosine

**Theorem**

*For arbitrary $\alpha, \beta \in \mathbb{R}$ we have*

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$  

\[
\begin{align*}
\cos(\alpha + \beta) &= \sin (\alpha + \beta + \frac{\pi}{2}) = \sin (\alpha + (\beta + \frac{\pi}{2})) \\
&= \sin \alpha \cos (\beta + \frac{\pi}{2}) + \cos \alpha \sin (\beta + \frac{\pi}{2}) \\
&= \sin \alpha \sin (\beta + \pi) + \cos \alpha \cos \beta \\
&= \cos \alpha \cos \beta - \sin \alpha \sin \beta.
\end{align*}
\]

Difference formulas

**Theorem**

*For arbitrary $\alpha, \beta \in \mathbb{R}$ we have*

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta,$$

*and*

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$  

\[
\begin{align*}
\text{We prove the first equation:} \\
\sin(\alpha - \beta) &= \sin (\alpha + (-\beta)) \\
&= \sin \alpha \cos (-\beta) + \cos \alpha \sin(-\beta) \\
&= \sin \alpha \cos \beta - \cos \alpha \sin \beta.
\end{align*}
\]
**Example**

*Find an exact value for* \( \cos \frac{\pi}{12} \).

- Write \( \frac{1}{12} = \frac{1}{3} - \frac{1}{4} \).

\[
\cos \left( \frac{\pi}{12} \right) = \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right)
= \cos \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right)
= \frac{1}{2} \cdot \frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{3} \cdot \frac{1}{2} \sqrt{2}
= \frac{\sqrt{2} + \sqrt{6}}{4}
\]

**Theorem**

*For arbitrary* \( \alpha \in \mathbb{R} \) *we have*

\[
\sin(2\alpha) = 2 \sin \alpha \cos \alpha,
\]

*and*

\[
\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha.
\]

- We prove the first equation:

\[
\sin(2\alpha) = \sin(\alpha + \alpha)
= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha
= 2 \sin \alpha \cos \alpha.
\]
**Theorem**

For arbitrary $\alpha \in \mathbb{R}$ we have

\[
\sin(2\alpha) = 2 \sin \alpha \cos \alpha,
\]

and

\[
\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha.
\]

- We prove the first equation:
  \[
  \sin(2\alpha) = \sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = 2 \sin \alpha \cos \alpha.
  \]
**Theorem**

For arbitrary $\alpha \in \mathbb{R}$ we have

$$\sin^2 \left( \frac{1}{2} \alpha \right) = \frac{1 - \cos \alpha}{2},$$

and

$$\cos^2 \left( \frac{1}{2} \alpha \right) = \frac{1 + \cos \alpha}{2}.$$

We prove the first equation. Let $\varphi = \frac{1}{2} \alpha$, then

$$\frac{1 - \cos \alpha}{2} = \frac{1 - \cos(2\varphi)}{2} = \frac{1 - (\cos^2 \varphi - \sin^2 \varphi)}{2}$$

$$= \frac{1 - \cos^2 \varphi}{2} + \frac{\sin^2 \varphi}{2} = \frac{\sin^2 \varphi}{2} + \frac{\sin^2 \varphi}{2}$$

$$= \sin^2 \varphi = \sin^2 \left( \frac{1}{2} \alpha \right).$$

**Example**

Find an exact value for $\cos \left( \frac{\pi}{12} \right)$.

- Use the half-formula for cosine:

  $$\cos^2 \left( \frac{\pi}{12} \right) = \cos^2 \left( \frac{1}{2} \cdot \frac{\pi}{6} \right) = \frac{1 + \cos \frac{\pi}{6}}{2} = \frac{1 + \frac{1}{2} \sqrt{3}}{2} = \frac{1}{2} + \frac{1}{4} \sqrt{3}.$$ 

- Hence $\cos \frac{\pi}{12} = \pm \sqrt{\frac{1}{2} + \frac{1}{4} \sqrt{3}}$.

- Since $\frac{\pi}{12}$ is an acute angle, the cosine of $\frac{\pi}{12}$ must be positive, so

  $$\cos \frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{1}{4} \sqrt{3}}$$
On previous slide: \( \cos \frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}} \),

on slide 40: \( \cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4} \).

Square both results:

\[
\left( \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}} \right)^2 = \frac{1}{2} + \frac{1}{4}\sqrt{3},
\]

\[
\left( \frac{\sqrt{2} + \sqrt{6}}{4} \right)^2 = \frac{2 + 6 + 2\sqrt{2}\sqrt{6}}{16} = \frac{8 + 4\sqrt{3}}{16} = \frac{1}{2} + \frac{1}{4}\sqrt{3}.
\]

\[\text{c}\]

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### Overview

| Periodicity | \( \sin(\alpha + 2\pi) = \sin \alpha \) and \( \sin(\alpha + \pi) = -\sin \alpha \)  

\( \cos(\alpha + 2\pi) = \cos \alpha \) and \( \cos(\alpha + \pi) = -\cos \alpha \) |
|----------------|--------------------------------------------------------------------------------------------------|
| Symmetry       | \( \sin(-\alpha) = -\sin \alpha \)  

\( \cos(-\alpha) = \cos \alpha \) |
| Congruence     | \( \sin \left( \alpha + \frac{\pi}{2} \right) = \cos \alpha \) and \( \sin \left( \alpha - \frac{\pi}{2} \right) = -\cos \alpha \)  

\( \cos \left( \alpha + \frac{\pi}{2} \right) = -\sin \alpha \) and \( \cos \left( \alpha - \frac{\pi}{2} \right) = \sin \alpha \) |
| Sum formulas   | \( \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \)  

\( \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \) |
| Difference formulas | \( \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \)  

\( \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \) |
| Doubling formulas | \( \sin(2\alpha) = 2 \sin \alpha \cos \alpha \)  

\( \sin^2 \left( \frac{1}{2}\alpha \right) = \frac{1}{2} - \frac{1}{2} \cos \alpha \)  

\( \cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha \)  

\( \cos^2 \left( \frac{1}{2}\alpha \right) = \frac{1}{2} + \frac{1}{2} \cos \alpha \) |
| Pythagoras’ thm | \( \cos^2 \alpha + \sin^2 \alpha = 1 \) |
Assignment: IMM1 - Tutorial 1.4.